

Features of ghost-gluon and ghost-quark bound states related to BRST quartets

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Abstract

The BRST quartet mechanism in infrared Landau gauge QCD is investigated. Based on the observed positivity violation for transverse gluons A_{tr} the field content of the non-perturbative BRST quartet generated by A_{tr} is derived. To identify the gluon's BRST-daughter state as well as the Faddeev-Popov-charge conjugated second parent state, a truncated Bethe-Salpeter equation for the gluon-ghost bound state in the adjoint colour representation is derived and studied. This equation is found to be compatible with the so-called scaling solutions of functional approaches. Repeating the same construction for quarks instead of A_{tr} leads in a similar way to a truncated Bethe-Salpeter equation for the quark-ghost bound state in the fundamental representation. Within the scaling solution the infrared divergence of the quark-gluon vertex is exactly the right one to make this Bethe-Salpeter equation infrared consistent.

Keywords: Yang-Mills theory, BRST, QCD, Confinement, Dyson-Schwinger equations, Bethe-Salpeter equation

1. Introduction

As it is well-known by now, gauge-fixing a Yang-Mills (YM) theory leaves nevertheless an important invariance: The YM action is invariant under a transformation which may be pictured as a gauge transformation but with the Faddeev-Popov ghost as transformation “parameter” [1, 2, 3]. This BRST transformation is on the perturbative level the corner stone in the proof of renormalizability, for an accurate discussion see *e.g.* Ref. [4]. In addition, it can be employed to define a physical state space which in this language is identified with the cohomology of the BRST charge operator, see Ref. [5] and references therein. Obviously, the definition of a positive-definite state space is intimately related to confinement: The coloured states underlying confinement cannot be attributed to asymptotic states in the S -matrix. To this end it is interesting to note that the BRST cohomology automatically contains only colour-singlet states.

These remarks make evident that the two purposes BRST cohomology serves require two different kinds of BRST multiplets.¹ There are on the one hand the perturbative BRST multiplets, first of all the elementary BRST quartet [5] which takes care of the cancellation of longitudinal and time-like gluons as well as ghosts and antighosts in all physical states. This is nothing else than the generalization of the Gupta-Bleuler formalism of QED. Or, as Weinberg [4] phrases it, the perturbative BRST quartet mechanism in YM is an $(N_c^2 - 1)$ -fold² duplication of the single cancellation mechanism in QED. For the definition of the state space as BRST cohomology these considerations are by far not sufficient. On the contrary, in QCD we expect the BRST cohomology to contain only glueballs and hadrons,

i.e. bound states. As bound states occur only beyond perturbation theory it becomes evident that one has to employ non-perturbative techniques in studying the BRST cohomology in general.

At this point it is interesting to note that recently a possibility has been suggested [6] to avoid the Neuberger 0/0 problem of lattice BRST [7] in a modified lattice Landau gauge which employs stereographic projections. Besides other fundamental issues elucidated in ref. [6] the following aspect is important in the present context: It is now clarified that the global BRST charge, necessary for the perturbative as well as for the non-perturbative quartet mechanism, can be well defined.

One interesting aspect of BRST multiplets follows from the nilpotency of the BRST transformation, see Sect. 2 for details. Every non-singlet state can then produce only one further state, making thus a doublet. It proves, however, useful to form quartets such that the Faddeev-Popov charge conjugated state of the daughter state in this doublet is used as a parent state which under BRST generates the 2nd daughter and thus completes the quartet. In this letter we will argue that positivity violation of transverse gluons imply that they are parent states. As we will show this implies that the 1st daughter has to be a gluon-ghost bound state, and the 2nd parent accordingly a gluon-antighost bound state. In Landau gauge which is ghost-antighost symmetric these two bound states become degenerate and are described by the same equation. Formally, the same remarks apply for the quartet generated by quarks. It is one of the main results to be presented here that the structures of the corresponding Bethe-Salpeter equations (truncated to keep the infrared leading terms) become very similar due to the dressing of the quark-gluon vertex.

In Sect. 2 we demonstrate briefly how to obtain the field content of the gluon- and the quark-generated non-perturbative

¹Besides the colourless BRST singlets there are only the so-called BRST quartets, see below.

² $N_c^2 - 1$ being the dimension of the adjoint representation.

BRST quartets.³ In Sect. 3 we discuss the ghost-gluon Bethe-Salpeter equation and provide arguments for a massless ghost-gluon bound state. In Sect. 4 the ghost-quark bound state equation is given and discussed. In Sect. 5 we conclude and close with an outlook.

2. BRST quartets in Landau gauge QCD

Throughout this letter we will only consider Landau gauge. To emphasize the nilpotency of the BRST transformation we will work in a representation with Nakanishi-Lautrup field B^a which becomes on-shell identical to the gauge fixing condition, $B^a = (1/\xi) \partial_\mu A_\mu^a$ where ξ is the gauge parameter of linear covariant gauges.

It is useful to picture the BRST transformation δ_B as a “gauge transformation” with a constant ghost field as parameter

$$\begin{aligned} \delta_B A_\mu^a &= \tilde{Z}_3 D_\mu^{ab} c^b \lambda, & \delta_B q &= -ig t^a \tilde{Z}_1 c^a q \lambda, \\ \delta_B c^a &= -\frac{g}{2} f^{abc} \tilde{Z}_1 c^b c^c \lambda, & \delta_B \bar{c}^a &= B^a \lambda, \\ \delta_B B^a &= 0, \end{aligned} \quad (1)$$

where D_μ^{ab} is the covariant derivative. The parameter λ lives in the Grassmann algebra of the ghost fields c^a , it carries ghost number $N_{\text{FP}} = -1$. \tilde{Z}_1 and \tilde{Z}_3 are the ghost-gluon-vertex and the ghost wave function renormalization constants. In Landau gauge one has $\tilde{Z}_1 = 1$.

Via the Noether theorem one may define a BRST charge operator Q_B which in turn generates a ghost number graded algebra on the fields, $\delta_B \Phi = \{iQ_B, \Phi\}$. Defining the ghost number operator Q_c one obtains

$$Q_B^2 = 0, \quad [iQ_c, Q_B] = Q_B. \quad (2)$$

This algebra is complete in the full (indefinite metric) state space of YM theory, resp., QCD. The BRST cohomology is then constructed as follows: The semi-definite physical subspace $\text{Ker } Q_B$ is defined on the basis of this algebra by those states which are annihilated by the BRST charge Q_B , $Q_B|\psi\rangle = 0$. Since $Q_B^2 = 0$, this subspace contains the space $\text{Im } Q_B$ of the so-called daughter states $Q_B|\phi\rangle$ which are images of their parent states in full state space. A physical Hilbert space is then obtained as the space of equivalence classes:

$$\mathcal{H}(Q_B) = \text{Ker } Q_B / \text{Im } Q_B. \quad (3)$$

This Hilbert space is isomorphic to the space of BRST singlets. All states are either BRST singlets or belong to quartets, this exhausts all possibilities. Note that the condition $Q_B|\psi\rangle = 0$ eliminates half of these metric partners from all S -matrix elements, leaving unpaired states of zero norm which do not contribute to any observable.

For constructing the so-called elementary quartet [9, 5] one considers the asymptotic states related to the time-like and longitudinal gluons as well as the ghost and the antighost.⁴ Hereby one gluon polarization and the antighost provide the parent states, the orthogonal gluon polarization and the ghost yield the daughter states. In all physical states the contribution of this quartet cancels similar to the cancellation of time-like and longitudinal photons in QED. This quartet is strictly perturbative in the sense that it also exists in the limit $g \rightarrow 0$. From the construction of this elementary quartet one can infer how to construct other perturbative “multi-particle” BRST quartets [5]: Starting from a state with negative norm (1st parent) one obtains the 1st daughter by acting with the BRST charge operator on the 1st parent. The Faddeev-Popov charge reflected state of the 1st daughter provides the 2nd parent. Acting on it with the BRST charge operator provides the 2nd daughter.

At the perturbative level the transverse gluons belong to the BRST cohomology which is, of course, in open conflict with the observed confinement of gluons. Therefore it has been speculated already decades ago that the transverse gluons are also part of a BRST quartet [9] which is then in turn believed to be an important aspect of gluon confinement. At approximately the same time it has been observed [11] that the antiscreening of gluons (which is a very welcome property as it explains asymptotic freedom) is already at the perturbative level in conflict with the positivity of the gluon spectral function. By now there is no doubt any more that the transverse gluons of Landau gauge QCD are positivity violating, see *e.g.* Ref. [12] and references therein. With the remarks given above this makes plain that “one-transverse-gluon” states are BRST parent states. Their respective daughters, however, cannot be the elementary “one-ghost” states because these are members of the elementary quartet. On inspecting eq. (1) one can immediately conclude that the corresponding daughter state needs to have the field content

$$\tilde{Z}_3 f^{abc} A_\mu^c c^b. \quad (4)$$

As for every “one-transverse-gluon” state there should occur exactly one daughter state this requires the existence of a ghost-gluon bound state in the adjoint representation. In this sense the resulting BRST quartet is strictly non-perturbative as the formation of bound states can be described only with non-perturbative techniques. The Faddeev-Popov charge reflected state is then an antighost-gluon bound state. Here Landau gauge provides an advantage as compared to general linear covariant gauges: In the limit $\xi \rightarrow 0$ the formalism becomes ghost-antighost-symmetric, and thus the existence of a ghost-gluon bound state implies the occurrence of a degenerate antighost-gluon bound state with same quantum numbers. Even having then the 2nd parent, the BRST transformation (1) leaves then three possibilities for the 2nd daughter: A ghost-antighost bound state, a ghost-antighost-gluon bound state, or a bound state of two differently polarized gluons. However, studying the 2nd daughter

³The main results of this section have been known, of course, since quite some time. However, in the way needed in the following it is not easily accessible in the literature. A more detailed description of the material summarized in Sect. 2 may be found in Ref. [8].

⁴As the Nakanishi-Lautrup field B^a relates to a linear combination of time-like and longitudinal gluons, the so-called backward polarization, the corresponding asymptotic “one-gluon” parent (daughter) states are forwardly (backwardly) polarized, see *e.g.* Chapter 16 of Ref. [10].

state is beyond the scope of this letter, and we will return below to the ghost-gluon bound state.

The respective issues for quarks in the Landau gauge are much less clear. First of all, it is not known whether quarks violate positivity. Although for light quarks dynamical chiral symmetry breaking (and for heavy quarks explicit chiral symmetry breaking) determines the infrared behaviour of the quark propagator the analytic structure of the quark propagator is highly sensitive to details in the quark-gluon vertex, see, e.g., Ref. [13]. The quark-gluon vertex for light quarks is, on the other hand, also very strongly influenced by dynamical chiral symmetry breaking [14, 15, 16]. Second, an inspection of eq. (1) reveals that if a BRST quartet is generated by quarks it can only be a non-perturbative one. Which mechanism then guarantees that the corresponding bound states are degenerate with the quark states is completely unknown.

Before trying to give an at least partial answer to this we will discuss the infrared behaviour of Landau gauge YM theory and the implications for a possible ghost-gluon bound state and its role in the corresponding BRST quartet.

3. Landau gauge YM theory and ghost-gluon bound state

Over the last decade the infrared behaviour of Landau gauge YM theory has been in the focus of many studies. Hereby it is interesting to note that in the deep infrared quite general statements can be deduced by employing functional equations. On the one hand, Dyson-Schwinger equations have been used to extend a previous analysis of gluon and ghost propagators [17, 18, 19, 20, 21] to all Yang-Mills vertex functions [22, 23]. Employing in addition the Function Renormalization Group Equations, and requiring that these two, seemingly different, towers of equations have to provide identical Green's functions, allows a powerful restriction on the type of the solution: There is one unique scaling solution with power laws for the Green's functions [24, 25] and a one-parameter family of solutions, the so-called decoupling solutions. These infrared trivial solutions possess as an endpoint exactly the scaling solution which is characterized by infrared power laws. Numerical solutions of the decoupling type (there called “massive solution”) have been published in [26, 27] and references therein. A recent detailed description and comparison of these two types of solutions has been given in Ref. [28], see also Refs. [29, 30, 31, 32].

Although almost all lattice calculations of the gluon propagator favor a decoupling solution lattice studies at strong coupling [33, 34, 35] reveal the existence of a regime where the scaling relation between the gluon and the ghost propagators is fulfilled, and the corresponding infrared exponent κ is very close to the value determined in truncated continuum studies with $\kappa = 0.595$. A potential explanation of these conflicting results has been discussed recently [36]: The infrared behaviour of the Green's functions may depend on the non-perturbative completion of the gauge.

In the present context it is important to note that the scaling solution respects BRST symmetry whereas every decoupling solution breaks it [28], although very likely only softly. In a strict sense our analysis as presented in the following will be

only valid if the scaling solution is a correct one. However, the situation is not as drastic as it may seem. First, if the conjecture of Ref. [36] is correct it is sufficient that only one non-perturbative completion of Landau gauge with scaling solution exists to make our analysis well-founded. Second, even if only decoupling type of solutions will turn out to be correct an extended BRST-like nilpotent symmetry is likely to take the role of the BRST symmetry [41], or the soft BRST symmetry breaking can be treated as spontaneous symmetry breaking [42], see also Ref. [43] and references therein. Note also that all our arguments about infrared dominance of diagrams will likely stay correct: The numerical value of a diagram which is infrared leading in the scaling solution will be enhanced (*i.e.* numerically large) in a decoupling solution if the latter is not far from the endpoint given by the scaling solution.

As a basis for the following discussion we will summarise the infrared behaviour of all one-particle irreducible Green's functions in the scaling solution in the simplified case with only one external spacelike scale $p^2 \rightarrow 0$. For a function with n external ghost and antighost as well as m gluon legs one has:

$$\Gamma^{n,m}(p^2) \sim (p^2)^{(n-m)\kappa}. \quad (5)$$

This solution fulfills all functional equations and all Slavnov-Taylor identities. It verifies the hypothesis of infrared ghost dominance [37] and leads to an infrared diverging ghost propagator as well as infrared diverging three- and four-gluon vertex functions.

As already stated gluons violate positivity [12, 13]. For the scaling solution this can be immediately deduced from the fact that for this solution the gluon propagator vanishes at zero virtuality, $p^2 = 0$. A further important property of the scaling solution is the infrared trivial behaviour of the ghost-gluon vertex which is in agreement with general arguments [38, 20].

To search for the anticipated ghost-gluon bound state we want to truncate the ghost-ghost-gluon-gluon scattering kernel to the infrared leading term. To this end we employ first the MATHEMATICA package DoDSE, resp., DoFun [39, 40] to derive the diagrammatic expressions for the one-particle irreducible ghost-ghost-gluon-gluon scattering kernel. The diagram-by-diagram infrared power counting⁵ verifies that in the scaling solution the infrared exponent is $-\kappa$. It also provides the infrared leading terms.

As two different kind of fields are involved there exists two distinct possibilities for the Dyson-Schwinger equation according to which leg one puts the bare vertex. Placing the bare vertex on a gluon line gives 56 diagrams (59 if quarks are included) with six (seven) of them containing 5-point or 6-point functions. Eleven of them are infrared leading. Assigning the bare vertex to a ghost leg leads to a Dyson-Schwinger equation with 13 diagrams on the r.h.s, one containing a 5-point function, and all of them infrared leading. As we have to truncate the system such to neglect $n \geq 5$ -point functions it is obvious that the second choice minimizes the truncation error.⁶ This is substantiated by

⁵Some more details of this analysis are given in ref. [8].

⁶Note that either neglecting or approximating the one-particle irreducible 5-

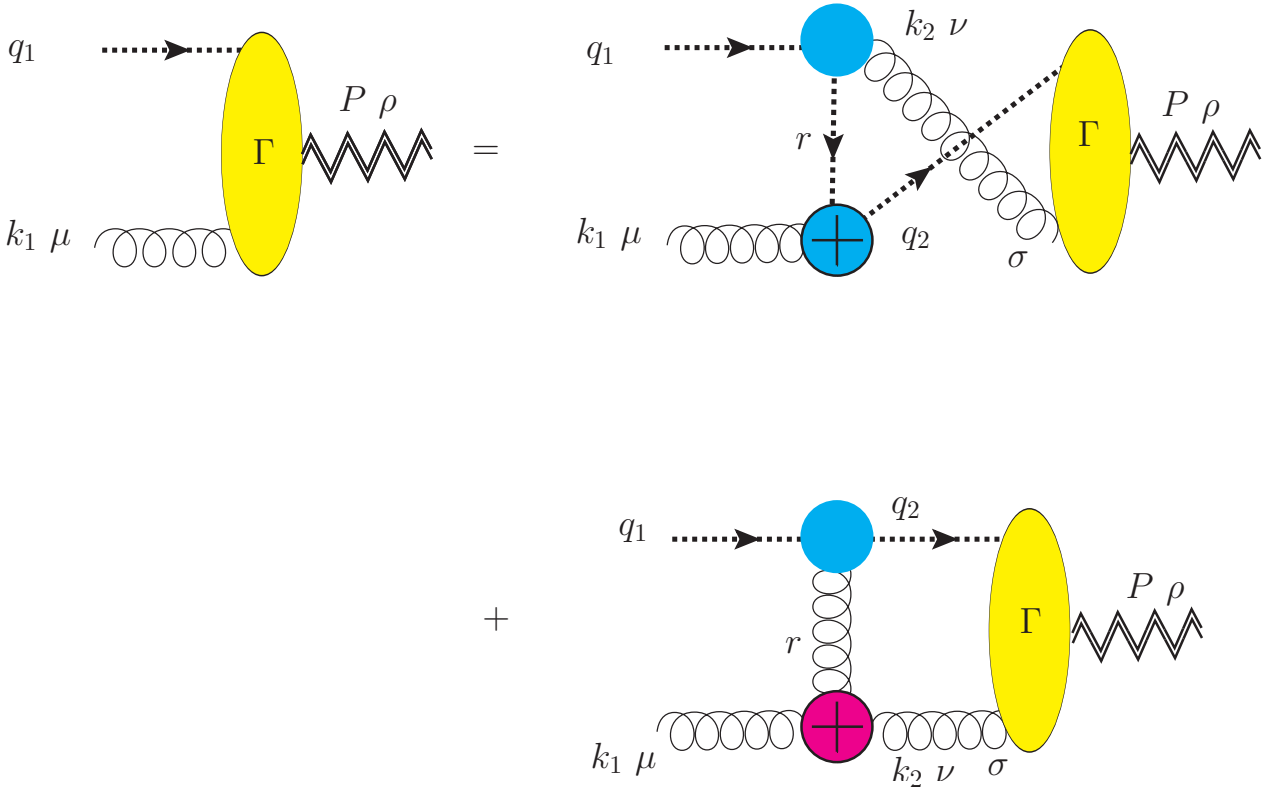


Figure 1: Graphical representation of the gluon-ghost Bethe-Salpeter equation. A cross denotes a dressed vertex.

the fact that the first choice would provide eventually a Bethe-Salpeter equation with one less term as if we will employ the second choice, see below.

The requirements for the diagrams on the r.h.s to be kept are: It should contain the one-particle irreducible ghost-ghost-gluon-gluon four-point function and no $n \geq 5$ -point function, it should be infrared leading, and it should contain the interaction in the ghost-gluon channel. This leaves two diagrams: (i) one with two ghost and one gluon propagator on internal lines. This is effectively a ghost exchange. (NB: A more precise description is that the leading tree-level diagram describes the splitting of the incoming gluon into a ghost-antighost pair and a fusion of the incoming ghost with the exchanged (anti-)ghost to a gluon, *cf.* the upper r.h.s of Fig. 1.) (ii) one with two gluon and one ghost propagators on internal lines. This is a gluon exchange. Note that this diagram is infrared leading because in the scaling solution the fully dressed three-gluon vertex is infrared divergent with an exponent -3κ . If we had chosen to put the bare vertex on the gluon leg (*cf.* the first choice above) this diagram would be infrared suppressed. We would also like to remark that from the 14 tensor structures of the three-gluon vertex at

least ten are infrared divergent with the same exponent [44].

Assuming the existence of a bound state as well as employing the usual decomposition of the (ghost-ghost-gluon-gluon) four-point function into Bethe-Salpeter amplitudes and performing the expansion around the pole (see *e.g.* Sect. 6.1 of Ref. [45]) one arrives at the Bethe-Salpeter equation depicted in Fig. 1. Using the propagator parameterizations of *e.g.* Ref. [13], the ghost-gluon vertex of Ref. [46], and the three-gluon vertex of Ref. [44] one can derive a self-consistent equation for the corresponding Bethe-Salpeter amplitude containing otherwise only known quantities. This leads, mostly due to the many tensor terms of the three-gluon vertex, to very lengthy expressions which will be given elsewhere. For the illustrational purposes of this letter we will neglect the gluon-exchange term and keep only the ghost exchange.

As stated above in Landau gauge the ghost-gluon vertex stays even in the scaling solution infrared trivial. Thus it is sufficient to consider only bare ghost-gluon vertices, *i.e.* it is sufficient to consider the ladder approximation to the first term of the above Bethe-Salpeter equation. (We also use $\tilde{Z}_1 = 1$ [38].) The explicit expression for the kernel, assuming as gauge group $SU(N_c)$ and working in Euclidean momentum space, reads then

$$\begin{aligned} \delta^{ab} H_{\mu\nu}(k_1, q_1; k_2, q_2) &= f^{acd} f^{bef} i g f^{cdg} (q_1 - k_2)_\nu \\ &\quad D_G(q_1 - k_2) i g f^{cfg} q_{2\mu} \\ &= -g^2 N_c^2 \delta^{ab} r_\nu q_{2\mu} D_G(r), \end{aligned} \quad (6)$$

where we have already taken care of the re-projection onto the

point function is necessary to obtain a solvable Bethe-Salpeter equation. However, as the diagram containing the 5-point function possesses the same infrared exponent as the other diagrams the resulting equation stays qualitatively correct, and it is reasonable to assume that it is quantitatively satisfactory. Please note also that the truncation is not based on the number of diagrams but based on the argument to keep as many infrared leading diagrams as technically possible.

adjoint representation and have used the momentum assignment as in the r.h.s of Fig. 1. Hereby $D_G(q_1 - k_2) = D_G(r)$ is the ghost propagator.

With this kernel and denoting the gluon propagator as $D_{\mu\nu}(k)$ we arrive at the gluon-ghost Bethe-Salpeter equation for bound state with four-momentum P

$$\Gamma_{\mu\rho}(k_1, q_1; P) = -\widetilde{Z}_3^2 g^2 N_c^2 \int \frac{d^4 r}{(2\pi)^4} r_\nu q_{2\mu} D_G(r) D_{\nu\sigma}(k_2) D_G(q_2) \Gamma_{\sigma\rho}(k_2, q_2; P). \quad (7)$$

As a side remark we want to mention that to arrive at the equation for the antighost-gluon bound state we only need to invert all ghost momenta and obtain therefore:

$$\widetilde{\Gamma}_{\mu\rho}(k_1, q_1; P) = -\widetilde{Z}_3^2 g^2 N_c^2 \int \frac{d^4 r}{(2\pi)^4} q_{1\nu} r_\mu D_G(r) D_{\nu\sigma}(k_2) D_G(q_2) \widetilde{\Gamma}_{\sigma\rho}(k_2, q_2; P). \quad (8)$$

As argued above we are looking for a massless bound state. This allows to specialise to the soft limit $P \rightarrow 0$. Employing the transversality of the Landau gauge gluon propagator,

$$D_{\nu\sigma}(k) = \left(\delta_{\nu\sigma} - \frac{k_\nu k_\sigma}{k^2} \right) \frac{Z(k^2)}{k^2} \quad (9)$$

it is straightforward to show that $\Gamma_{\mu\rho}(k, -k; 0)$ (resp., $\widetilde{\Gamma}_{\mu\rho}(k, k; 0)$) is transverse, too:

$$\Gamma_{\mu\rho}(k, -k; 0) =: \left(\delta_{\mu\rho} - \frac{k_\mu k_\rho}{k^2} \right) F(k^2). \quad (10)$$

Defining the ghost renormalization function $G(q^2) = q^2 D_G(q^2)$ one obtains then from eq. (7)

$$F(k_1^2) = \widetilde{Z}_3^2 g^2 N_c^2 \int \frac{d^4 k_2}{(2\pi)^4} \frac{G((k_1 + k_2)^2)}{(k_1 + k_2)^2} \frac{G(k_2^2)}{k_2^2} \frac{Z(k_2^2)}{k_2^2} \frac{1}{3} (k_1 \cdot k_2) \left(1 - \frac{(k_1 \cdot k_2)^2}{k_1^2 k_2^2} \right) F(k_2^2). \quad (11)$$

It is an easy exercise to verify that the Bethe-Salpeter equation (8) provides exactly the same equation as (11) and therefore the degeneracy of the massless antighost-gluon bound state if the massless gluon-ghost bound state exists.

The numerical solution of eq. (11) is beyond the scope of the present letter, it will be presented elsewhere. Here we only want to mention that, first, $F(k^2)$ has the peculiar feature $F(0) = 0$, and, second, due to truncation errors we do not expect the equation to be exactly fulfilled. As, however, we have kept the infrared leading term it is important to test whether there is no contradiction on the analytical level.

For the scaling solution we have the power laws

$$Z(p^2) \sim (p^2)^{2\kappa}, \quad G(p^2) \sim (p^2)^{-\kappa}, \quad (12)$$

which implies that the kernel of eq. (11) possesses a vanishing anomalous infrared exponent and is independent of the infrared exponent κ . We therefore get the expected result that the Bethe-Salpeter equation for a massless (anti-)ghost-gluon bound state

is consistent within the infrared analysis of the scaling solution of functional equations. Furthermore, as the combination $\alpha^{gh}(p^2) = \frac{g^2}{4\pi} G^2(p^2) Z(p^2)$ has the form of a running coupling with an infrared fixed point $\alpha^{gh}(0) = 8.92/N_c$ (see *e.g.* Sect. 2.3 of Ref. [47]) it is plausible that the kernel of eq. (11) is strong enough to lead to a bound state. All these findings corroborates the validity of the employed method, and therefore we will investigate the quark-gluon bound state with the same method.

4. Quark-gluon bound state

The scaling solution for the YM Green's functions leads to dynamical chiral symmetry breaking in the quark sector [16]. The quark propagator which is of the general form

$$S(p) = \frac{i\not{p} + M(p^2)}{p^2 + M^2(p^2)} Z_f(p^2) \quad (13)$$

is then infrared finite, $S(0) = Z_f(0)/M(0)$. The twelve possible Dirac tensor structures of the quark-gluon vertex are then all infrared divergent with an infrared exponent $-\kappa - 1/2$. As a side remark we want to add that the same infrared divergence results for vanishing gluon momentum, and that this leads to $1/k^4$ behaviour of the kernel in the four-quark function, k being the momentum exchange. This is indicative of a linearly rising potential between static quarks, and thus quark confinement.

Furthermore, the Slavnov-Taylor identities require that the ghost-ghost-quark-quark scattering kernel is infrared trivial, see Sect. 3.9 in Ref. [16]. This information is absolutely necessary for the following analysis.

As in the previous subsection we have two choices for the Dyson-Schwinger equation according to which leg one puts the bare vertex. Both choices lead to seven diagrams on the r.h.s of the equation, in both choices there appears one diagram with a 5-point function (ghost-ghost-quark-quark-gluon). However, choosing the quark leg to carry the bare vertex only one diagram is infrared leading, namely exactly the one with the 5-point function. Thus truncating the equation to $n \leq 4$ -point functions would lead to a contradiction. Choosing a ghost leg to place the bare vertex there are four infrared leading diagrams: One with the 5-point function, and three according to the three possible (*s, t, u*) interaction channels. Therefore, employing the same truncation requirements and the same derivation of the Bethe-Salpeter equation as in the previous subsection one arrives at the equation depicted in Fig. 2. This equation is in full agreement with the infrared analysis of the scaling solution, *i.e.* it is a valid bound state equation, and in its kernel the infrared exponent κ cancels.

Employing all twelve tensor structures of the quark-gluon vertex does generate a very lengthy equation. Again we will restrict for illustrational purposes to the perturbatively leading tensor component,

$$\Gamma_\mu^a(p_1, p_2; k) \rightarrow -ig \frac{\lambda^a}{2} \gamma_\mu V(p_1, p_2; k) \quad (14)$$

where V diverges like $(k^2)^{-\kappa-1/2}$ for $k^2 \rightarrow 0$. Using the result

$$\frac{\lambda^a}{2} \frac{\lambda^c}{2} \frac{\lambda^b}{2} f^{abc} = -\frac{i}{4} (N_c^2 - 1) \quad (15)$$

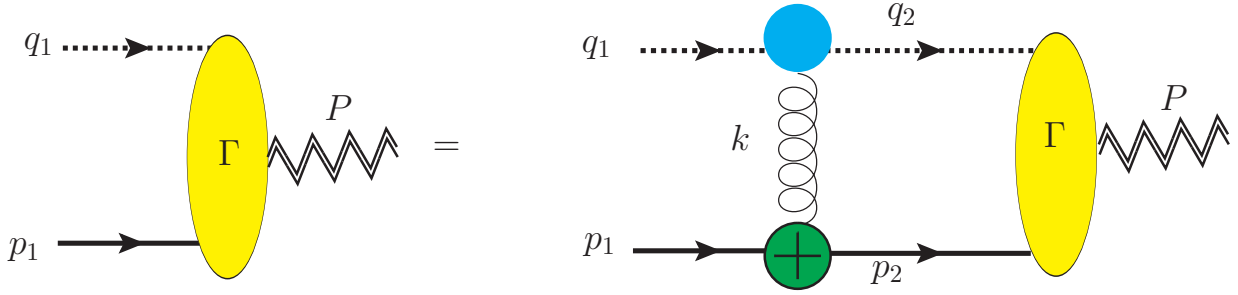


Figure 2: Graphical representation of the quark-ghost Bethe-Salpeter equation. Note that the quark-gluon vertex is fully dressed.

valid for $SU(N_c)$ one can project the bound state onto the fundamental colour representation. The Bethe-Salpeter equation for the ghost-quark bound state amplitude with total momentum P reads then:

$$\Gamma(p_1, q_1; P) = \frac{i}{4}(N_c^2 - 1) \int \frac{d^4 k}{(2\pi)^4} q_{2\mu} D_{\mu\nu}(k) \gamma_\nu \quad (16)$$

$$V(p_1, p_2; k) S(p_2) D_G(q_2) \Gamma(p_2, q_2; P)$$

with $p_2 = p_1 + k$ and $q_2 = q_1 - k$. At first sight it may look like that the Bethe-Salpeter amplitude of this scalar-fermion bound state has four independent amplitudes. This is, however, not correct. Should the bound state occur at a non-vanishing $P^2 = M_B^2 < 0$ (to avoid saying the “ghost-quark bound state is massive”) there will exist two degenerate solutions with respectively positive and negative energy. Then it is sufficient to study only the positive-energy solution by explicitly putting a positive-energy projector Λ^+ into the tensor decomposition⁷

$$\Gamma(p, q; P) = \left(H'_1 + \frac{i}{M_B} (\not{p} - \not{q}) H'_2 \right) \Lambda^+, \quad \Lambda^+ = \frac{1}{2} \left(1 + \frac{\not{P}}{iM_B} \right). \quad (17)$$

The Dirac algebra to generate the set of coupled equations for H'_1 and H'_2 is lengthy but straightforward. However, as the equations themselves are somewhat vast we refrain from displaying them here.

Again we want to simplify for illustrational purposes and simply set $\Gamma(p, q; P) = H(p, q; P)$, *i.e.* to a scalar function. In the case of the dominance of the “upper Dirac component” this should be a reasonable approximation. To further demonstrate the properties of the kernel we make the simplest possible choice for $V(p_1, p_2; k)$ agreeing with its infrared limit: $V(p_1, p_2; k) \rightarrow G(k^2)/\sqrt{k^2}$. The corresponding equation for the simplified amplitude $H(p, q; P)$ reads then:

$$H(p_1, q_1; P) = g^2 \frac{N_c^2 - 1}{4} \int \frac{d^4 k_2}{(2\pi)^4} \frac{Z(k^2)}{k^2} \frac{G(k^2)}{\sqrt{k^2}} \frac{G(q_2^2)}{q_2^2}$$

$$\frac{Z_f(p_2^2)}{p_2^2 + M^2(p_2^2)} \left(q_1 \cdot p_1 - \frac{q_1 \cdot P_1 p_1 \cdot p_2}{k^2} \right) H(p_2, q_2; P). \quad (18)$$

⁷This is very similar to the bound state equation of a scalar diquark and a quark for a nucleon, see *e.g.* Sect. 7.3 of Ref. [45].

The structural similarities to eq. (11) are striking. Again it is likely that the kernel is strong enough to lead to a bound state. Providing further evidence that these bound states are degenerate with quark states will, however, require the study of the system with the full ansatz (17).

At the end of this section we want to add a cautionary remark: Although the statements given in this section do not hold only in the quenched limit of QCD but also for QCD with dynamical fermions, the resulting equations cannot be used directly. Note that *e.g.* in the step from eq. (16) to eq. (18) a result for the quark-gluon vertex [16] has been used which is only known in the quenched limit.

5. Conclusions and outlook

The observed positivity violation of the Landau gauge gluon propagator imply that, if BRST is an unbroken symmetry, the transverse gluons generate a non-perturbative BRST quartet. The other three members are bound states. For two of these we have derived a (truncated) bound state equation by analysing the one-particle-irreducible ghost-ghost-gluon-gluon four-point function. The corresponding Bethe-Salpeter equation containing in its kernel infrared divergent Green’s function is infrared consistent, a result which is far from being trivial. In addition, we have provided arguments that the kernel is strong enough such that a bound state is formed. All these findings corroborate the existence of the non-perturbative BRST quartet with transverse gluons as first parent states. On the other hand, one might also argue that the employed method is consistent and can therefore be extended to study the analogue problem for quarks.

With the same method we derive then the bound state equation for a quark-ghost bound state truncated to the infrared leading term. We note that this equation is within the scaling solution only then infrared consistent if the recently found infrared divergence of the fully dressed quark-gluon vertex is taken into account. We take this as further evidence that the infrared behaviour of the quark-gluon vertex is intimately related to the issue of quark confinement. Nevertheless, further studies of the quark-gluon vertex are highly desirable to generate progress on the question whether in Landau gauge QCD there is a non-perturbative BRST quartet generated by quarks.

The numerical solution of the gluon-ghost bound state equation (11) can be done with standard techniques, it will be published elsewhere. In order to solve numerically the quark-ghost bound state equation (16) and its analogue for the quark-antighost bound state the full Dirac tensor ansatz (17) has to be plugged in, and the corresponding Dirac algebra has to be performed. However, although the resulting set of equations are lengthy they can be solved with methods which are by now standard.

Here we did not touch on the question of the respective second BRST daughters. Following the same arguments as employed here the corresponding tasks should be doable. However, before starting an investigation in this direction it might be worthwhile to repeat the presented investigation with Functional Renormalization Group Equations to minimize further or even eliminate truncation errors.

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